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AUTHOR(S):

前田, 英敏

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# ADJOINT BUNDLES OF AMPLE AND SPANNED VECTOR BUNDLES ON ALGEBRAIC SURFACES

HIDETOSHI MAEDA

## 0. Introduction

Here we carry out a brief survey of [Lanteri-Maeda 91].

The linear system  $|K_X + C|$  “adjoint” to a curve  $C$  on a surface  $X$  has played an important role in understanding the geometry of  $X$  since the early days of surface theory. The adjoint bundle  $K_X + L$  to a very ample line bundle  $L$  on a smooth complex projective surface  $X$  was investigated in modern terms by Sommese [Sommese 79] and Van de Ven [Van de Ven 79]. The study of  $K_X + L$  was made in [Lanteri-Palleschi 84] when  $L$  is simply supposed to be an ample line bundle. Recently, several authors ([Fujita 90], [Wiśniewski 89], [Ye-Zhang 90]) have dealt with a generalized polarized pair  $(X, \mathcal{E})$  consisting of a smooth complex projective variety  $X$  and an ample vector bundle  $\mathcal{E}$  on  $X$ , and have investigated the nefness and the ampleness of the adjoint line bundle  $K_X + \det \mathcal{E}$ . In this note we treat an ample and spanned vector bundle  $\mathcal{E}$  of rank  $r (r \geq 2)$  on a smooth complex projective surface  $X$ , and study some properties of the adjoint bundle  $K_X + \det \mathcal{E}$ . Precisely, we ask the following

### Questions.

- (a) When is  $K_X + \det \mathcal{E}$  spanned ?

(b) When is  $K_X + \det \mathcal{E}$  very ample ?

We can obtain a complete answer to (a) by using Reider's method [Reider 88]. In fact, we will prove the

**Theorem A.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a smooth complex projective surface  $X$ . Set  $L = \det \mathcal{E}$ . Then  $K_X + L$  is spanned unless  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$ .*

The same method also enables us to give a partial but satisfactory answer to (b). The precise statement of our result is as follows:

**Theorem B.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a smooth complex projective surface  $X$ . Set  $L = \det \mathcal{E}$  and assume  $L^2 \geq 9$ . Then  $K_X + L$  is very ample unless  $(X, \mathcal{E})$  is one of the following.*

- (1)  $X$  is a  $\mathbf{P}^1$ -bundle over a smooth curve  $C$  and  $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$  for any fiber  $F$  of  $X \rightarrow C$ .
- (2)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 3})$ .
- (3)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1))$ .
- (4)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, T_{\mathbf{P}})$ .

Note that this theorem proves the 2-dimensional part of the conjecture (2.6) in [Lanteri-Pallesi-Sommese 89] since  $L^2 = 9$  in the three cases (2), (3) and (4). By the way we notice that the higher dimensional part of it should be restated in the following form.

**Conjecture.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $n \geq 3$*

on a smooth projective variety  $X$  of dimension  $n$ . Let  $L = \det \mathcal{E}$  and assume  $L^n \geq (n+1)^n + 1$ . Then  $K_X + L$  is very ample unless  $X$  is a  $\mathbf{P}^{n-1}$ -bundle over a smooth curve  $C$  and  $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus n}$  for any fiber  $F$  of  $X \rightarrow C$ .

In case  $L^2 \leq 8$ , we use the adjunction theory developed by Sommese and Van de Ven [Sommese-Van de Ven 87] to make an answer to (b) on the assumption that  $L$  is very ample. Our result is the

**Theorem C.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a smooth complex projective surface  $X$ . Set  $L = \det \mathcal{E}$ . Assume that  $L$  is a very ample line bundle with  $L^2 \leq 8$ . Then  $K_X + L$  is very ample unless  $(X, \mathcal{E})$  is one of the following.*

- (1)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$ .
- (2)  $X \cong Q^2$ , a smooth hyperquadric in  $\mathbf{P}^3$ , and  $\mathcal{E} \cong \mathcal{O}_Q(1)^{\oplus 2}$ .
- (3)  $X \cong \mathbf{P}_C(\mathcal{F})$  and  $\mathcal{E} \cong \rho^* \mathcal{G} \otimes H(\mathcal{F})$  for some indecomposable vector bundles  $\mathcal{F}$  and  $\mathcal{G}$  of rank two on an elliptic curve  $C$  with  $c_1(\mathcal{F}) = c_1(\mathcal{G}) = 1$ , where  $H(\mathcal{F})$  is the tautological line bundle on  $X$  and  $\rho$  is the projection  $X \rightarrow C$ .
- (4)  $X$  is a Del Pezzo surface with  $K_X^2 = 2$ , and  $\mathcal{E} \cong (-K_X)^{\oplus 2}$ .
- (5)  $X$  is as in case (4) and  $\mathcal{E} \cong f^* \mathcal{F} \otimes (-K_X)$ , where  $f : X \rightarrow \mathbf{P}^2$  is the blowing-up of  $\mathbf{P}^2$  along seven points and  $\mathcal{F}$  is the cokernel of a bundle monomorphism  $\mathcal{O}_{\mathbf{P}}(-1)^{\oplus 2} \rightarrow (\Omega_{\mathbf{P}}^1 \otimes \mathcal{O}_{\mathbf{P}}(1))^{\oplus 2}$ .

We will work over the complex number field. Basically we use the standard notation from algebraic geometry. For a vector bundle  $\mathcal{E}$  on  $X$ , the tautological line bundle on the projective space bundle  $\mathbf{P}_X(\mathcal{E})$  associated to  $\mathcal{E}$  is denoted by

$H(\mathcal{E})$ . A vector bundle is called *spanned* if it is generated by its global sections.

## 1. Preliminaries

This note relies heavily on Reider's method, which we recall first in the following form.

**Lemma 1.** [Reider 88] *Let  $N$  be a nef line bundle on a smooth projective surface  $X$ .*

(1) *If  $N^2 \geq 5$  and  $K_X + N$  is not spanned, then there exists an effective divisor  $E$  satisfying either*

$$NE = 0, E^2 = -1 \text{ or } NE = 1, E^2 = 0.$$

(2) *If  $N^2 \geq 9$  and  $K_X + N$  is not very ample, then there exists an effective divisor  $E$  satisfying one of the following conditions.*

$$NE = 0, E^2 = -1 \text{ or } -2;$$

$$NE = 1, E^2 = 0 \text{ or } -1;$$

$$NE = 2, E^2 = 0;$$

$$N \equiv 3E, E^2 = 1.$$

Second we use Wiśniewski's idea [Wiśniewski 89, Lemma 3.2] to obtain a result on ample and spanned vector bundles on curves.

**Lemma 2.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a projective curve  $C$ . Take arbitrary points  $p_1, p_2, \dots, p_{r-1}$  of  $C$  with  $\mu_i = \text{mult}_{p_i}(C)$ .*

(1) *If  $C$  is rational, then  $c_1(\mathcal{E}) \geq (\sum_{i=1}^{r-1} \mu_i) + 1$ .*

(2) If  $C$  is non-rational, then  $c_1(\mathcal{E}) \geq (\sum_{i=1}^{r-1} \mu_i) + 2$ .

**Corollary 1.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a projective variety  $X$ . Put  $L = \det \mathcal{E}$ . Then  $X$  has no effective 1-cycles  $E$  such that  $LE < r$ .*

**Corollary 2.** *Let  $X, \mathcal{E}$  and  $L$  be as above. If an effective 1-cycle  $E$  on  $X$  satisfies  $LE = r$ , then  $E \cong \mathbf{P}^1$ .*

We need also the following lemma.

**Lemma 3.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a smooth projective variety  $X$  of dimension  $n \geq 2$ . Then  $H(\mathcal{E})^{n+r-1} \geq 3$ .*

Furthermore we can prove a slight strengthening of Wiśniewski's theorem [Wiśniewski 89, Theorem 3.4] which will be used later on.

**Lemma 4.** *Let  $X$  be a smooth projective variety of dimension  $n \geq 1$  and  $\mathcal{E}$  an ample and spanned vector bundle on  $X$  of rank  $r \geq n$ . Assume  $c_n(\mathcal{E}) = 1$ . Then  $(X, \mathcal{E}) \cong (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}}(1)^{\oplus n})$ .*

## 2. Proof of Theorem A

**Theorem A.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a smooth projective surface  $X$ . Set  $L = \det \mathcal{E}$ . Then  $K_X + L$  is spanned unless  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$ .*

*Proof.* Assume that  $(X, \mathcal{E}) \not\cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 2})$ . Then we have  $c_2(\mathcal{E}) \geq 2$  by

Lemma 4, since  $c_2(\mathcal{E}) > 0$  by [Bloch-Gieseker 71]. Combining the formula  $L^2 = c_2(\mathcal{E}) + H(\mathcal{E})^{r+1}$  with Lemma 3 gives  $L^2 \geq 5$ , so that Lemma 1 applies; but the exceptions to the spannedness of  $K_X + L$  are excluded in view of Corollary 1, and we are done. Q.E.D.

### 3. Proof of Theorem B

**Theorem B.** *Let  $\mathcal{E}$  be an ample and spanned vector bundle of rank  $r \geq 2$  on a smooth projective surface  $X$ . Set  $L = \det \mathcal{E}$  and assume  $L^2 \geq 9$ . Then  $K_X + L$  is very ample unless  $(X, \mathcal{E})$  is one of the following.*

- (1)  $X$  is a  $\mathbf{P}^1$ -bundle over a smooth curve  $C$  and  $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$  for any fiber  $F$  of  $X \rightarrow C$ .
- (2)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1)^{\oplus 3})$ .
- (3)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1))$ .
- (4)  $(X, \mathcal{E}) \cong (\mathbf{P}^2, T_{\mathbf{P}})$ .

*Proof.* (outline) Assume that  $K_X + L$  is not very ample. Then by Lemma 1 and Corollary 1, there exists an effective divisor  $E$  satisfying one of the following.

- (i)  $LE = 2, E^2 = 0$ ;
- (ii)  $L \equiv 3E, E^2 = 1$ .

(3.1) In case (i), combining  $LE = 2$  with Corollary 1 and Corollary 2 gives  $r = 2$  and  $E \cong \mathbf{P}^1$ . Since  $E^2 = 0$ ,  $X$  is ruled and  $E$  is a fiber of the ruling. We use Corollary 1 again to see that every fiber  $F$  is irreducible and reduced. Thus

$X$  is a  $\mathbf{P}^1$ -bundle over a smooth curve  $C$  and  $\mathcal{E}_F \cong \mathcal{O}_F(1)^{\oplus 2}$ .

(3.2) In case (ii),  $E$  is ample and so  $E$  is irreducible and reduced. By Corollary 1  $LE = 3$  implies  $r \leq 3$ . If  $r = 3$ , then from Corollary 2,  $E \cong \mathbf{P}^1$ . By the classification theory of polarized surfaces of sectional genus zero [Lanteri-Palleschi 84, Corollary 2.3], we have two possibilities:

$$(3.2.1) \ (X, \mathcal{O}_X(E)) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(i)), i = 1, 2.$$

$$(3.2.2) \ (X, \mathcal{O}_X(E)) \text{ is a scroll over } \mathbf{P}^1.$$

In case (3.2.1),  $i = 1$  and  $L = \mathcal{O}_{\mathbf{P}}(3)$ . Consider the vector bundle  $\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}}(-1)$ . This is trivial when restricted to any line in  $\mathbf{P}^2$ . Therefore itself is trivial, and hence  $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}}(1)^{\oplus 3}$ . In case (3.2.2), we may assume  $X = \mathbf{P}_{\mathbf{P}}(\mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(-e))$  for some  $e \geq 0$ . Thus  $E$  is very ample. Since  $E^2 = 1$ , we have  $(X, \mathcal{O}_X(E)) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(1))$ . This is absurd.

(3.3) In the following we can assume  $r = 2$ . Then we can prove that the arithmetic genus  $g(E) \leq 1$ . Therefore the classification theory of polarized surfaces of sectional genus  $\leq 1$  applies.

(3.4) Now suppose  $g(E) = 0$ . Then the same argument as in (3.2) shows  $(X, L) \cong (\mathbf{P}^2, \mathcal{O}_{\mathbf{P}}(3))$ , hence  $\mathcal{E}$  is a uniform bundle of splitting type  $(2, 1)$ . By the classification theory of uniform bundles on  $\mathbf{P}^2$  [Van de Ven 72],  $\mathcal{E}$  is either the direct sum of two line bundles or the twisted tangent bundle. Consequently  $\mathcal{E} \cong \mathcal{O}_{\mathbf{P}}(2) \oplus \mathcal{O}_{\mathbf{P}}(1)$  or  $T_{\mathbf{P}}$ .

(3.5) To complete the proof of Theorem B, we discuss the case  $g(E) = 1$ . There are two possibilities [Lanteri-Palleschi 84, Corollary 2.4]:

$$(3.5.1) \ X \text{ is a Del Pezzo surface and } \mathcal{O}_X(E) = -K_X.$$



(3.5.2)  $(X, \mathcal{O}_X(E))$  is a scroll over an elliptic curve  $C$ .

In case (3.5.1),  $K_X^2 = 1$  and  $L = -3K_X$ . In case (3.5.2), we can write  $X = \mathbf{P}_C(\mathcal{F})$  for some *normalized* vector bundle  $\mathcal{F}$  of rank two on  $C$ . Moreover,  $\mathcal{O}_X(E) = H(\mathcal{F}) + \rho^*A$  for some line bundle  $A$  on  $C$ , where  $\rho$  is the projection. Set  $e = -c_1(\mathcal{F})$  and  $a = \deg A$ . Then  $e \geq -1$  and  $E^2 = 2a - e = 1$ . By the criterion for an ample line bundle, we have  $e = -1$  and  $a = 0$ . Thus  $\mathcal{F}$  is indecomposable and  $L = 3H(\mathcal{F}) + \rho^*B$  for some line bundle  $B$  of degree 0 on  $C$ . In sum,  $(X, L)$  is one of the following:

- (1)  $X$  is a Del Pezzo surface with  $K_X^2 = 1$ , and  $L = -3K_X$ .
- (2)  $X \cong \mathbf{P}_C(\mathcal{F})$  for some indecomposable vector bundle  $\mathcal{F}$  of rank two on an elliptic curve  $C$  with  $c_1(\mathcal{F}) = 1$ .  $L = 3H(\mathcal{F}) + \rho^*B$  for some line bundle  $B$  of degree 0 on  $C$ , where  $\rho$  is the projection  $X \rightarrow C$ .

However, we can show that neither (1) nor (2) occurs. Q.E.D.

For the proof of Theorem C, we refer to [Lanteri-Maeda 91].

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Hidetoshi Maeda

Department of Mathematics

School of Education

Waseda University

1-6-1 Nishi-Waseda, Shinjuku-ku

Tokyo 169-50

Japan